

Predictive Model of Undrained Shear Strength Based on Random Field Theory

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ABSTRACT

A model for estimating the undrained shear strength is developed for a mangrove swamp near-shore site in Niger Delta Nigeria. The site is underlain by an upper layer of sensitive soil mixed with organic matter followed by alternating layers of clay and sand. The development of the model is based on the random field theory using cone penetration test (CPT) data from the geotechnical site investigation report. Data from 16 CPTs, arranged in a 4x4 grid and spaced at 50m between test holes, were selected for the study with intermediate CPT data simulated between adjacent test holes using a pseudo-averaging technique to reduce the spacing to 12.5m. Data simulation is justified since the soil profile displayed uniform and continuous soil grouping and sequencing over the entire length and breadth of the field. A 2-dimensional spatial variability model for predicting the value of undrained shear strength at any depth across the entire field is formulated, using the statistical quantities estimated for the soil layers. The model estimates the undrained shear strength within a margin of error $\pm 15\%$.

Keywords - Cone Penetration Test, Random field theory, spatial variability, undrained shear strength.

INTRODUCTION

Spatial variability in soil property has always been recognized in geotechnical engineering practice and for a comprehensive understanding of the subject, it is possible to carry out a detailed characterization of the spatial variation in any direction with sufficiently high number of measurements which will be considered unnecessary given the large number of samples and time required for such exercise. The hypothesis of randomness is usually adopted in practice to bridge the gap in knowledge with most recent research efforts resorting to statistical and probabilistic methods in characterizing spatial variability [1]. These methods have evolved from simple statistical description of the soil property to the more intricate random field theory. The inadequacy of simple statistical description was highlighted by [2] with reference to the second-moment statistics of mean and standard deviation which alone are incapable of describing the spatial variation of soil properties, since two sets of measurements with similar second-moment statistics and statistical distribution may exhibit differences in spatial distribution.

It is now generally accepted that the statistical description of the soil property in terms of the mean, variance, and scale of fluctuation, θ , is adequate to fully characterize the spatial variability of a soil property. In practice, the spatial variability of the soil property is estimated from samples obtained from a population and the analysis usually performed using either: (a) random field theory – a variation of time series analysis [3] and [4]; or (b) geostatistics [5]. A key requirement in using both analytical methods is that of data stationarity which assumes the invariance of the statistics of a data set to spatial location [2]. In a more recent study, [6] proposed a method of site variability characterization that applied knowledge of spatial statistics to quantify site variability indices using Cone Penetration Test (CPT) data.

In this paper, a 2-dimensional variability model for predicting the value of undrained shear strength at any depth across the entire field is developed by combining the classical random field theory and the method for evaluating horizontal spatial variability as proposed by [6].

RANDOM FIELD THEORY

Spatial Variability

The random field model of spatial variability assumes spatial dependence such that the soil properties $X(x_1)$ and $X(x_2)$ exhibit some form of dependence that decreases with separation distance. The interdependence of the properties at points in the field is characterized using joint bivariate distribution $f_{x_1, x_2}(x_1, x_2)$. However, for three or more points, the complete probabilistic description of the random process becomes complex and difficult to use in practice. The characterization problem is, nevertheless, simplified by assuming a Gaussian process and stationarity of data which allows the complete joint distribution to be quantified by the mean vector and covariance matrix, and makes the distribution independent on spatial position but dependent only on relative positions of points [7]. Assumption of stationarity implies that the statistical properties of the random field remain the same when the spatial origin changes position.

To simplify the application of random field theory, a transformation of the variables by decomposition is often performed to convert the non-stationary field to a stationary or nearly stationary field. The decomposition transformation technique idealizes the soil property as comprising of a deterministic trend component and a fluctuating or variable component expressed in form of an additive equation [2] and [8]:

$$\psi(z) = t(z) + \xi(z) \quad (1)$$

The objective in the decomposition process is primarily to obtain an estimate and remove the deterministic component, $t(z)$, while ensuring that the residual random component, $\xi(z)$, remains stationary. Analysis of inherent variability involves modeling the residual component of the soil property by statistical means. The residual component is assumed to have a spatial structure defined by the scale of fluctuation, θ , and the autocovariance function, $C(\tau)$, where τ is the distance between observation points [9].

The modelling of the soil parameters relies much on two essential statistical properties of the random field namely: autocovariance, c_k , and autocorrelation coefficient, ρ_k , at lag k . In practice, c_k and ρ_k are estimated from the samples obtained from a population. The sample autocovariance c_k^* and the sample autocorrelation coefficient, at lag k , r_k are defined as follows [10]:

$$c_k^* = \frac{1}{n} \sum_{i=1}^{n-k} (X_i - \bar{X})(X_{i+k} - \bar{X}) \quad (2)$$

and

$$r_k = \frac{c_k^*}{c_0} = \frac{\sum_{i=1}^{n-k} (X_i - \bar{X})(X_{i+k} - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} \quad (3)$$

\bar{X} = average of the observations X_1, X_2, \dots, X_n , and $0 \leq k \leq n$

The plot or graph of c_k^* for lags $k=0, 1, 2 \dots$ represents the sample autocovariance function (ACVF), while the plot of r_k for lags $k=0, 1, 2 \dots K$ represents the sample autocorrelation function (ACF), where K is the maximum number of lags for r_k calculations (e.g., $K=n/4$). In practice, estimation of θ is done by fitting the theoretical correlation (Table I) to the sample autocorrelation function [2], [9], and [11].

The autocorrelation between two locations separated by horizontal and vertical distances, Δh and Δz respectively, in a three-dimensional zero-mean random field $\xi(x,y,z)$, can be estimated with Equation (4), where ξ is the residual or detrended property field and (x,y,z) is the spatial location, with x and y being the horizontal coordinates and z the depth coordinate [12].

$$\rho(\Delta h, \Delta z) = \frac{\text{Cov}[\xi(x,y,z), \xi(x+\Delta x, y+\Delta y, z+\Delta z)]}{\sqrt{\text{Var}[\xi(x,y,z)]} \times \sqrt{\text{Var}[\xi(x+\Delta x, y+\Delta y, z+\Delta z)]}} \quad (4)$$

where $\text{var}(\cdot)$ denotes variance: $\text{cov}(\cdot)$ denotes covariance, $\Delta h = (\Delta x^2 + \Delta y^2)^{0.5}$ is the horizontal separation distance.

The modelling of the spatial variability of geotechnical material requires a minimum of three parameters: the mean, μ ; a measure of variance, σ^2 (standard deviation, σ , or coefficient of variation); and the scale of fluctuation, θ , that associates the correlation of properties with distance [13]. Large values of θ , for a particular property, signifies that the property slowly fluctuates with distance about the mean, suggesting a more continuous deposit, while a small θ is an indication of the property fluctuating rapidly about the mean, suggesting a more randomly varying material [10].

A. Stationarity or Statistical Homogeneity

Statistical homogeneity means that the entire joint probability density function of soil property values, at an arbitrary number of locations within the soil unit, remains the same even as location changes. From the physical perspective, statistical homogeneity or stationarity is attributed to data from uniform soil material that passed through similar geological processes. If the soil profiles are improperly demarcated into statistically homogeneous (stationary) sections, it will result in biased estimate of the variance of the soil data. It is therefore important to ensure that the entire soil profile within the zone of influence is divided into number of statistically homogeneous or stationary sections, and the data within each layer subjected separately to further statistical analysis [14].

A random field with non-stationary mean and variance can always be converted to a weakly stationary field by linear transformation using Equation (5) [7].

Table I: Some common correlation models

Correlation Model	Expression	Scale of Fluctuation θ
Simple exponential	$\rho(\tau) = \exp[- \tau /b]$	$2b$
Gaussian exponential	$\rho(\tau) = \exp\{-\pi[\tau /c]^2\}$	$\sqrt{\pi}c$
Second-order autoregressive process	$\rho(\tau) = \exp^{- \tau /d}(1 + \tau /d)$	$4d$
Cosine exponential	$\rho(\tau) = \exp^{- \tau /\alpha} \cos(\tau/\alpha)$	α

Source: Lloret-Cabot et al. (2013)

$$X'(t) = \frac{X(t) - \mu(t)}{\sigma(t)} \quad (5)$$

With transformation, the random field $X'(t)$ will now have zero mean and unit variance everywhere in the field.

B. Site Variability Assessment

The method of site variability characterization by [6] applied knowledge of spatial statistics to quantify site variability indices using CPT data. The method estimated the vertical variability index (VVI) and the horizontal variability index (HVI).

The VVI is comprised of three components: (!) intra-layer variability index, which calculates the vertical

variability of the soil profile based on variations in the cone tip resistance (q_c) and skin friction (f_s) using the scale of fluctuation, (2) log variability index captures the vertical variability of the soil profile based on the number of soil types present, and (3) cone resistance vertical variability index that captures the variability as a result of the presence of extreme dissimilar soil layers in the soil profile.

The HVI estimates horizontal variability based on the geotechnical parameter correlation across CPT soundings in a domain. The CPT soundings were considered in pairs and for each pair, a measure of difference between the trends of the parameter with depth is computed followed by the determination of the cross-correlation coefficient using Equation (4). A high cross-correlation value and small trend difference of a CPT pair is an indication of strong correlation and similarity between the two CPTs, suggesting low variability in the horizontal direction of the field.

II. METHODOLOGY

Data from a geotechnical site investigation report of a refinery project located in a mangrove swampy near-shore site in Niger Delta region of Nigeria was used for the study. From a total of 96 data sets available, 20 data sets (16 CPT and 4 borehole data), were carefully selected to present equally spaced CPT grid, suited for random field theory application. The inferred undrained shear strength, s_u , from the CPT data were used for the analyses.

A. Method of Data Analysis

The soil profile generated from the borehole log and CPT data identified five distinct soil layers. To ensure statistical homogeneity or stationarity within the domain, for a seamless application of the random field theory, the entire soil profile within the zone of influence was divided into number of statistically homogeneous or stationary sections, and the data within each layer subjected separately to statistical analysis [14].

Data from each CPT test hole was evaluated to determine the value of geotechnical parameter at the different strata of the soil profile. The analyses of data from each CPT hole were performed using the following steps [10]:

- a) Examine the data of the parameter across the depth and transform the non-stationary data into stationary data. Where the data exhibited a trend, decomposition was required. The ordinary least square (OLS) method was used to estimate the trend.
- b) Normalize the detrended data (residual) by dividing the residuals by the corresponding standard deviation to produce a standard normal field ($\mu = 0, \sigma = 1$)
- c) Confirm stationarity of the residuals using Kendall's τ test.
- d) Calculate sample autocovariance and autocorrelation functions using Equations (2) and (3) respectively.
- e) Estimate the vertical scale of fluctuation or correlation length, θ , by fitting a theoretical model from Table I to the plot of sample ACF over lag distance
- f) Calculate the Bartlett's distance (i.e., distance over which the samples are autocorrelated).

Formulation of the random field model describing the variation of a soil parameter across the field was performed by defining two key parameters to account for the vertical and the horizontal variability of the soil property. The expected values of the soil parameter at mid points of the soil layer expressed as a function of the vertical correlation and the horizontal variability index (HVI), expressed as a function of distance, provided a model that described the variation of the soil parameter across the field or domain. The model therefore comprised of two components - vertical variability, and horizontal variability components.

The covariance matrix decomposition method was used to model the vertical variability while the procedure described in [6] was used to model the horizontal variability of the soil property. For purpose of this study, only the undrained shear strength was modeled.

III. RESULTS AND DISCUSSION

A. Soil Profile Generation

The CPT data was analyzed with the aid of [15], a software, which generates soil profiles using the Robertson 1986, Robertson 1990 SBT and Jefferies & Been 2006 charts. Comparative examination of the soil behavior type in Fig 1, and the CPT data shows that some of the soil layers are mixture of “clays” and “sands”. The soils were classified into major soil groups using a three-step approach that allowed thin layers to be merged into neighboring layers [16].

- SBT chart band approach – merging thin layers into adjacent layers by consideration of the secondary soil type(s) classification
- Soil group approach – merging thin layers into adjacent layers of the same soil group
- Average q_c approach – merging thin layers into adjacent layers with similar average q_c

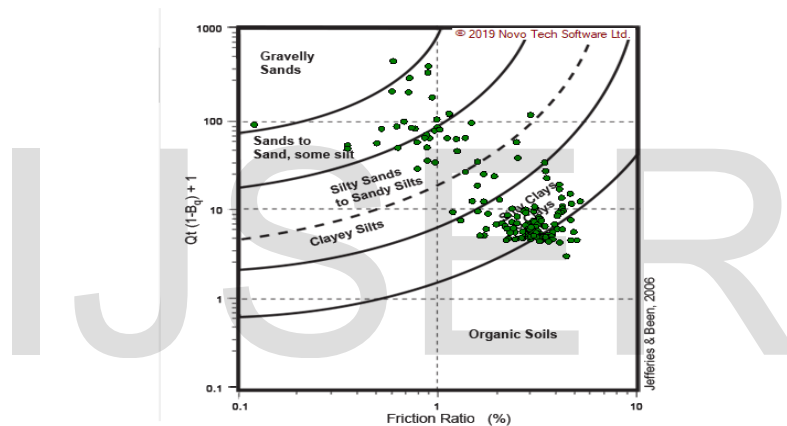


Fig 1: Jefferies & Been 2006 Soil type

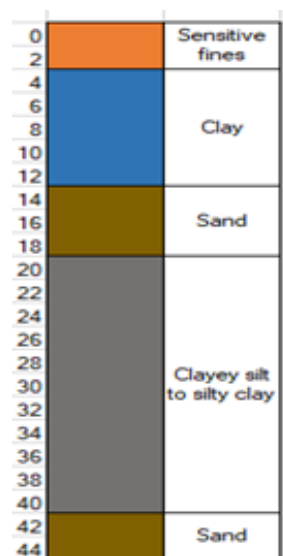


Fig 2: Adjusted Soil Profile

The final soil profile generated, shown in Fig 2, comprised of five main soil groups.

B. Variability Analyses of Undrained Shear Strength

The random field modeling of vertical variability of undrained shear strength is illustrated using data from CP14. Trend removal was achieved in the clay units with linear regression lines. The cosine exponential correlation model gave a close fit to the sample ACF. The bartlett's limits, defining the distance of autocorrelation, were estimated for each soil layer. Figs 3 and 4 show the fitting of the theoretical and sample ACFs over the lag distance while Table II presents the summary of results from the analyses. The correlation lengths of undrained shear strength from results of analyses of other CPT data are presented in Table III. The results show that the coefficient of variability (COV) of the correlation length is high at the upper sensitive soil layer but decreases as depth increases an indication that the value of the soil parameter is more predictable at greater depths

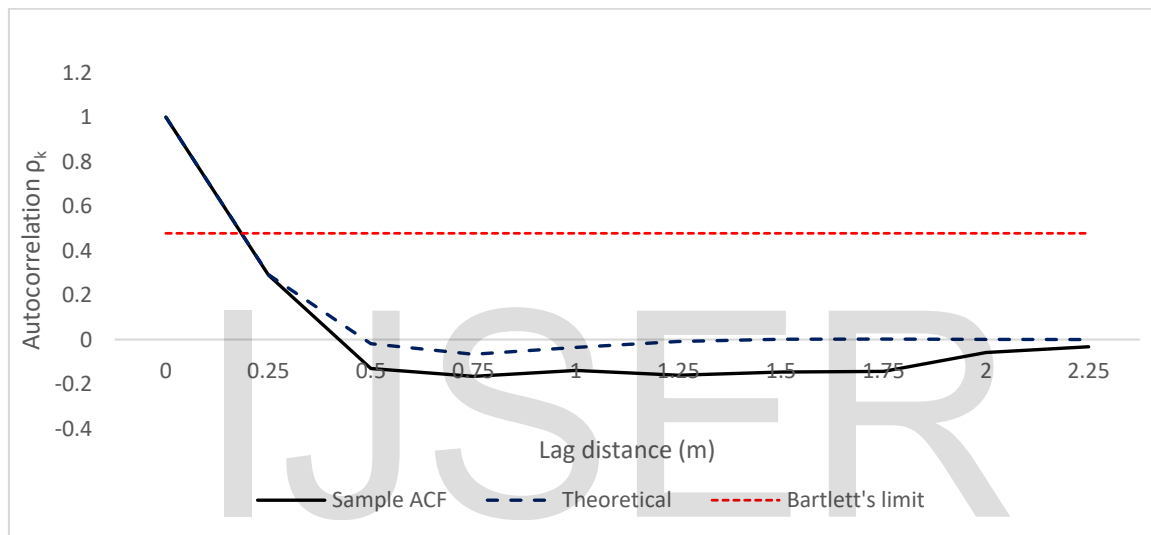


Fig 3: Sample and theoretical ACF at depth 0.25-5.25m (sensitive fine)

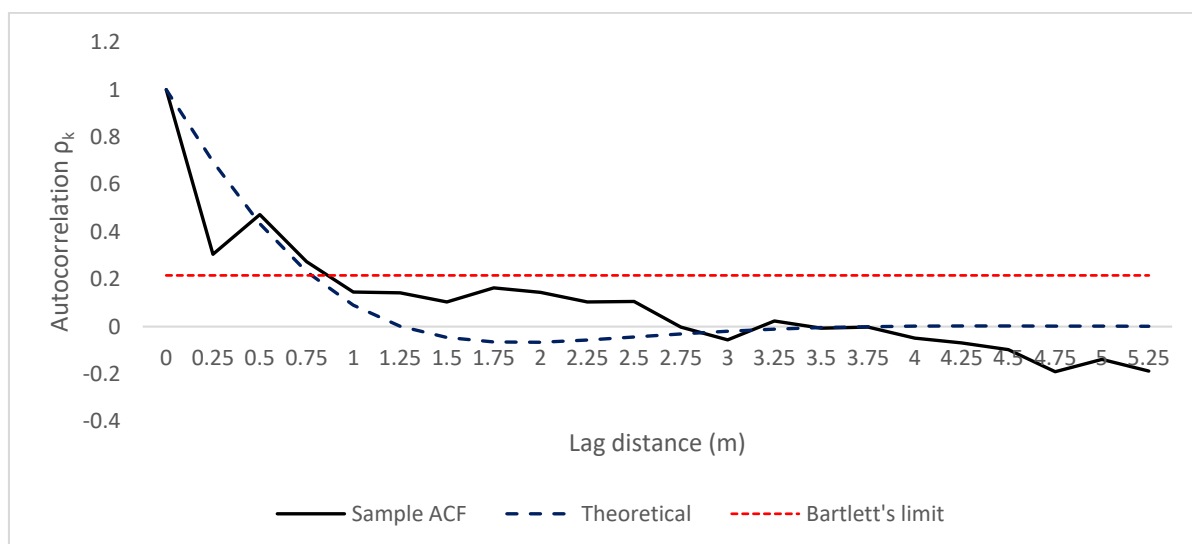


Fig 4: Sample and theoretical ACF at depth 16.25-37.25m (clayey silt to silty clay)

Table II: Summary of vertical variability analyses of s_u

Depth range of stratum (m)	Value of Parameter (m)	Bartlett's distance(m)	Scale of Fluctuation, θ (m)
0.25-5.25	$\alpha=0.300$	± 0.428	0.191
6.25-12.25	$\alpha=0.400$	± 0.392	0.388
17.00-37.25	$\alpha=0.800$	± 0.216	0.875

Table III: Vertical scale of fluctuation, θ_v , of s_u

Sensitive fines	0.25 - 5.25m		
Clay		6.25 – 12.25m	16.5 – 37.5m
Variable	θ_v (m)	θ_v (m)	θ_v (m)
CP1	0.320	0.782	0.871
CP2	0.278	0.690	0.631
CP3	0.465	0.585	0.678
CP4	0.348	0.356	0.503
CP5	0.386	0.638	0.671
CP6	0.320	0.650	0.652
CP7	0.405	0.420	0.517
CP8	0.482	0.316	0.518
CP9	0.434	0.628	0.613
CP10	0.238	0.706	0.601
CP11	0.682	0.601	0.830
CP12	0.204	0.354	0.667
CP13	0.328	0.468	0.768
CP14	0.191	0.388	0.875
CP15	0.510	0.875	0.500
CP16	0.805	0.708	0.773
Mean (m)	0.400	0.573	0.667
Std Dev (m)	0.166	0.169	0.127
COV (%)	41.45	29.54	19.02

The horizontal variability indices were determined over a limiting distance of 12m and the average HVI for the clay units at different depths are presented in Table IV.

Table IV: Horizontal Variability Indices of clay soil units

Average depth range (m)	Average HVI values
0.25 – 5.25	0.572
6.25 – 12.25	0.655
16.5 – 37.5	0.509

C. Random Field Model of the Undrained Shear Strength

Using the covariance matrix decomposition method with the 4x4 matrices of vertical correlation structure of the soil layers, a random field estimation of the values of the undrained shear strength were obtained. A power function trendline of the variation of the undrained shear strength across the domain provided the regression coefficients of the vertical variability model. Table V shows the random field estimates of the undrained shear strength, while the vertical variability models are presented in Table VI for the different layers of clay.

The horizontal variability indices were determined over a limiting distance of 12m and the power function trendline of the plot of the indices over the distance provided the regression coefficients of the horizontal variability model. Table VII presents the horizontal variability model for the site.

Table V: Estimated undrained shear strength s_u :

Sensitive fines	0.25 - 5.25m		
Clay			
Variable	s_u (kPa)	s_u (kPa)	s_u (kPa)
CP1	28.89	29.69	73.84
CP2	16.43	46.73	61.87
CP3	28.19	30.30	92.30
CP4	16.99	37.31	84.87
CP5	19.57	49.89	69.43
CP6	15.89	31.16	63.13
CP7	29.75	28.38	71.68
CP8	18.00	39.26	81.84
CP9	16.84	44.84	63.82
CP10	16.01	33.69	69.19
CP11	25.33	33.16	75.64
CP12	13.96	47.66	79.00
CP13	13.57	38.96	63.82
CP14	15.98	38.59	64.17
CP15	20.48	26.43	63.74
CP16	16.81	30.44	69.36
Mean (m)	19.54	36.66	71.73
Std Dev (m)	5.41	7.45	8.96
COV (%)	27.7	20.3	12.5

Table VI: Vertical Variability model of s_u

Average depth range (m)	Vertical variability s_{uv} , component	Typical α value
0.25 – 5.25	$s_{uv} = 15.2\alpha^{-0.15}$	0.400
6.25 – 12.25	$s_{uv} = 31.2\alpha^{-0.004}$	0.573
16.5 – 37.5	$s_{uv} = 64.8\alpha^{-0.039}$	0.667

Table VII: Horizontal Variability model of s_u

Average depth range (m)	Horizontal variability s_{uh} , component	Typical V_h values
0.25 – 5.25	$s_{uh} = V_h \cdot h^{-0.30}$	0.572
6.25 – 12.25	$s_{uh} = V_h \cdot h^{-0.20}$	0.655
16.5 – 37.5	$s_{uh} = V_h \cdot h^{-0.33}$	0.509

The random field model for the undrained shear strength of the soil is derived by combining the two variability components.

$$s_{u(avg)} = s_{uv} + s_{uh} \quad (6)$$

The random field model of the undrained shear strength at any depth across the field is generally expressed as

$$s_{u_i(avg)} = m\alpha_i^{\beta_i} + V_{hi}h^{\delta_i} \quad (7)$$

m is the mathematical expected value of the undrained shear strength at depth z , α is a variable linked to the vertical variability of the soil property (mean correlation length), β is regression coefficients obtained by fitting a power trendline on the plot of the mid depth undrained shear strength against the CP test holes at different depths of occurrence of the soil layer; V_{hi} is a variable linked to the horizontal variability of the soil parameter across the domain (average horizontal variability index of the soil layer); h is the distance at which the s_u is being measured and δ is the coefficient of the power fit of the trendline of plot of HVI over a maximum distance of 12m.

For the field studied the typical values of the coefficients in the random field model of the undrained shear strength at the different depths of occurrence of clay soil are presented in Table VIII

Table VIII: Coefficients for s_u spatial variability model

Average depth range (m)	m	α_i	β_i	V_{hi}	δ_i
0.25 – 5.25	15	0.40	-0.150	0.572	-0.30
6.25 – 12.25	31	0.57	-0.004	0.655	-0.20
16.5 – 37.5	65	0.67	-0.039	0.509	-0.33

The model was validated by comparing the predicted values with the field data. The model predicted the undrained shear strength values with error margin of $\pm 15\%$. Fig 8 compares the field measured and model predicted undrained shear strength.

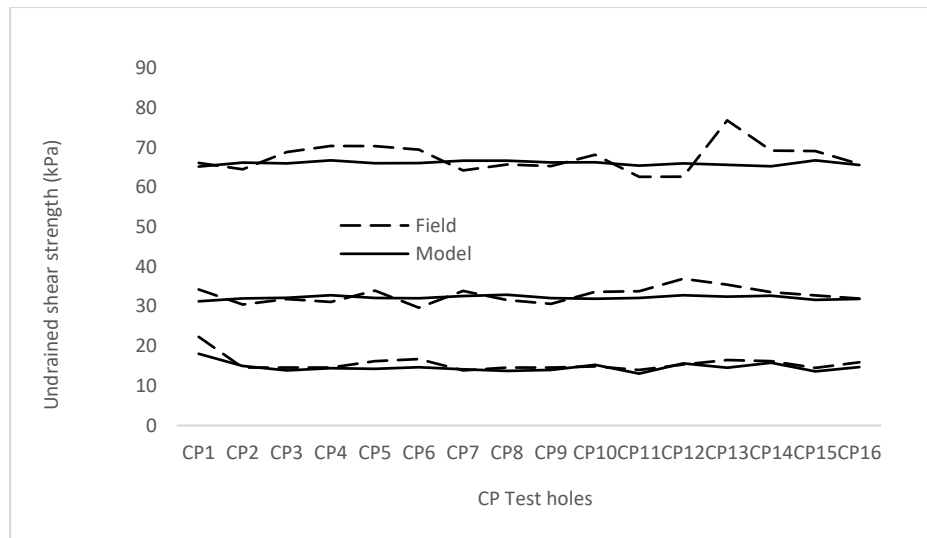


Fig 8: Comparison of Field measured and model predicted s_u

CONCLUSIONS

Given the correlation length and the expected value of the undrained shear strength of the clay soil units, it is possible to estimate the undrained shear strength of the field using the random field model (Equation (7)). Correlation lengths can be assumed with confidence for preliminary designs based on the analysis of data from the database.

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